GAYATRI VIDYA PARISHAD COLLEGE OF ENGINEERING FOR WOMEN

(AUTONOMOUS)

(Affiliated to Andhra University, Visakhapatnam)

B.Tech. - I Semester Regular Examinations, December / January – 2025

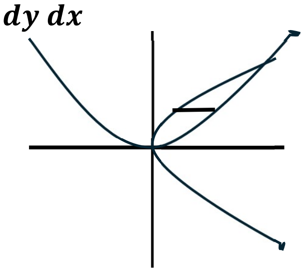
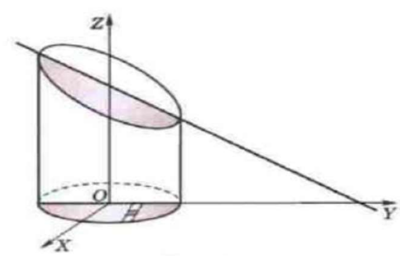
CALCULUS AND DIFFERENTIAL EQUATIONS

(Common to All branches)

SCHEME OF VALUATION

Q No	Question
1(a)	**If $u = (x^2 + y^2 + z^2)^{-1/2}$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$** $\frac{\partial u}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$ -----------------------------→ 2M $\frac{\partial^2 u}{\partial x^2} = \frac{(2x^2-y^2-z^2)}{(x^2+y^2+z^2)^{5/2}}$ -----------------------------→ 2M similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{(2y^2-x^2-z^2)}{(x^2+y^2+z^2)^{5/2}}, \frac{\partial^2 u}{\partial z^2} = \frac{(2z^2-y^2-x^2)}{(x^2+y^2+z^2)^{5/2}}$ -----------------------------→ 2M Therefore $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$ -----------------------------→ 1M
1(b)	**Calculate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$, if $u = x^2 - 2y, v = x + y + z, w = x - 2y + 3z$.** $u_x = 2x, u_y = -2, u_z = 0;\ v_x = 1, v_y = 1, v_z = 1; w_x = 1, w_y = -2, w_z = 3$ -----→ 3M $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$ -----------------------------→ 2M $= 10x + 4$ -----------------------------→ 2M
2(a)	**If $u = sin^{-1}(x - y), x = 3t$ and $y = 4t^2$, show that $\frac{du}{dt} = 3(1 - t^2)^{-1/2}$.** $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ -----------------------------→ 1M $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}, \frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}}$ -----------------------------→ 2M $\frac{dx}{dt} = 3, \frac{dy}{dt} = 12t$ -----------------------------→ 1M $\frac{du}{dt} = \frac{3-12t^2}{\sqrt{1-(3t-4t^2)^2}}$ -----------------------------→ 1M $= \frac{3-12t^2}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$ -----------------------------→ 2M
2(b)	**Determine the Taylor's series expansion of $f(x, y) = x^2 + 3y^2 - 9x - 9y + 26$ about the point (2, 2).** Given $f = x^2 + 3y^2 - 9x - 9y + 26,\ f(2,2) = 6$ $f_x = 2x - 9$; $f_x(2,\ 2) = -5$ $f_y = 6y - 9$; $f_y(2,\ 2) = 3$ $f_{xx} = 2, f_{xy} = 0, f_{yy} = 6$ } -------------------→ 3M Taylor's series expansion of $f(x, y)$ about the point $(a,\ b)$ -------------------→ 2M $f(x, y) = 6 - 5(x - 2) + 3(y - 2) + (x - 2)^2 + 3(y - 2)^2$ -------------------→ 2M

3(a)	**Discuss the maxima and minima of $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.** Given $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ $f_x = 4x^3 - 4x + 4y$, $f_y = 4y^3 + 4x - 4y$ --------------------→ 1M $f_x = 0,\ f_y = 0 \implies x = -y$ --------------------→ 1M The stationary points are (0, 0), $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ --------------------→ 1M $r = f_{xx} = 12x^2 - 4,\ s = f_{xy} = 4, t = f_{yy} = 12y^2 - 4$ --------------------→ 1M $rt - s^2 = 0$. It needs further investigation --------------------→ 1M $rt - s^2 > 0$ *and* $r > 0$ at $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, so f attains its minimum at both $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ and the minimum value is -8 --------------------→ 2M
3(b)	**Discuss the maxima, minima of the function $f = x^2 + y^2 + z^2$ where x, y, z are connected by the relation $xyz = 216$.** The Lagrangean function is $F = f + \lambda\phi = x^2 + y^2 + z^2 + \lambda(xyz - 216)$ -----------→ 1M $f_x = 0, f_y = 0,\ f_z = 0 \implies 2x + \lambda yz = 0,\ 2y + \lambda xz = 0,\ 2z + \lambda xy = 0$ -----------→ 3M $\therefore\ \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy} = -\frac{\lambda}{2}$ ------------→ 1M $x^2 = y^2 = z^2 \implies x = y = z$ ------------→ 1M $\therefore x = y = z = 6$ ------------→ 1M Hence the function attains it extremum at (6, 6, 6)
4(a)	**Examine the function $x^3 + y^3 - 3axy$ for the maxima and minima.** Let $f = x^3 + y^3 - 3axy$ $f_x = 0, f_y = 0 \implies 3x^2 - 3ay = 0$ *and* $3y^2 - 3ax = 0$ ------------→ 1M Solving above equations we get $x = y$ ------------→ 1M The stationary points are (0, 0), (a, a) ------------→ 1M $r = f_{xx} = 6x,\ s = f_{xy} = -3a, t = f_{yy} = 6y$ ------------→ 1M At (0, 0), $rt - s^2 = -9a^2 < 0$, f attains neither its minimum nor its maximum at (0, 0), so it is a saddle point ------------→ 1M At (a, a) $rt - s^2 = 27a^2 > 0$ If $a > 0, r > 0$ so f attains its minimum at (a, a) ------------→ 1M If $a < 0, r < 0$ so f attains its maximum at (a, a) ------------→ 1M
4 (b)	The Lagrangean function is $F = f + \lambda\phi = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$ ------→ 1M $f_x = 0, f_y = 0,\ f_z = 0 \implies 2x + a\lambda = 0,\ 2y + b\lambda = 0,\ 2z + c\lambda = 0$ -----------→ 3M $\therefore\ \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = -\frac{\lambda}{2}$ ------------→ 1M $\therefore x = \frac{ap}{a^2+b^2+c^2}$, $y = \frac{bp}{a^2+b^2+c^2}$, $z = \frac{cp}{a^2+b^2+c^2}$ ------------→ 1M The minimum value of the function is $\frac{p^2}{a^2+b^2+c^2}$ ------------→ 1M
5(a)	**Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dxdydz$.** $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dxdydz = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dydx$ ------------→ 1M $= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy(1 - x^2 - y^2) dydx$ ------------→ 1M $= \frac{1}{8}\int_0^1 x(1 - x^2)^2\ dx$ ------------→ 3M $= \frac{1}{48}$ ------------→ 2M

5(b)	**By applying the change of order of Integration evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$**  Over the given region x varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$ y varies from 0 to $4a$ --------→ 2M +1 M (Including diagram) $\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dy\,dx = \int_0^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dy$ ----------→ 2 M $= \frac{16a^2}{3}$ ----------→ 2 M
6(a)	**Evaluate $\int_0^5\int_0^{x^2} x(x^2+y^2)dxdy$** $\int_0^5\int_0^{x^2} x(x^2+y^2)dxdy = \int_0^5\int_0^{x^2}(x^3+xy^2)dydx$ ------------→ 1M $= \int_0^5\left(x^5+\frac{x^7}{3}\right)dx$ ------------→ 3M $= 5^6\left(\frac{29}{24}\right)$ ------------→ 3M
6(b)	**Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.** The required volume is V = $\iint_R z\,dx\,dy$ ---------------→ 1M where R is the projection of the surface on xy-plane  $= \int_{-2}^{-2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\,dx\,dy$ ---------------→ 2M $= 2\int_{-2}^{-2}\int_0^{\sqrt{4-y^2}}(4-y)\,dx\,dy$ ---------------→ 1M $= 8\times 2\int_0^2\sqrt{4-y^2}\ dy$ ---------------→ 2M $= 16\pi$ ---------------→ 1M
7(a)	**Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$.** Given equation is $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$ $\frac{\partial M}{\partial y}=x^2-4xy,\ \frac{\partial N}{\partial x}=6xy-3x^2$ $\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ so given equation is not exact ---------------→ 2M $I.F = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$ ---------------→ 1M Multiplying given equation by I.F on both sides weget $\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(\frac{3}{y}-\frac{x}{y^2}\right)dy=0$ ---------------→ 1M G.S is $\int_{y\ const} M_1\,dx+\int(terms\ of\ N_1\ not\ containing\ x)\ dy=c$ ---------------→ 1M $\frac{x}{y}+ln\left(\frac{y^3}{x^2}\right)=c$ ---------------→ 2M

7(b)	**Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$** $y_c = (c_1e^x + c_2xe^x) = c_1u + c_2v$ -----------------→ 1 M Where $u = e^x$; $v = xe^x$ $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = e^{2x}$ -----------------→ 2 M $y_p = Ae^x + Bxe^x$ $A = -\int \frac{vR}{w}dx;$ $B = \int \frac{uR}{w}dx$ -----------------→ 1 M A $= -x$; $B = \ln x$ -----------------→ 2 M $\therefore$ $y_p = -xe^x + (\ln x)xe^x$ Hence the GS is $y = y_c + y_p = (c_1e^x + c_2xe^x) - xe^x + (\ln x)xe^x$ -----------------→ 1 M
8 (a)	**Solve $(D^2 - 1)y = e^x + x^2e^x$** $y_c = (c_1e^x + c_2e^{-x})$ where c_1, c_2 are arbitrary constants -----------------→ 2 M $y_p = \frac{1}{D^2-1}e^x + \frac{1}{D^2-1}e^x x^2 = I_1 + I_2$ $I_1 = \frac{1}{D^2-1}e^x = \frac{xe^x}{2D} = \frac{xe^x}{2}$ -----------------→ 2 M $I_2 = \frac{1}{D^2-1}e^x x^2 = \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2} - \frac{1}{4}\right]$ -----------------→ 2 M G.S is $y = y_c + y_p = c_1e^x + c_2e^{-x} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2} - \frac{1}{4}\right]$ -----------------→ 1 M
8(b)	**A body originally at $80^o c$ cools down to 60^oc in 20 minutes, the temperature of the air being 40^oc. What will be the temperature of the body after 40 minutes from the original?** Statement: Newtons Law of cooling -----------------→ 1 M $\theta = \theta_0 + ce^{-kt}$ -----------------→ 1 M $c = 40, k = -\frac{1}{20}ln\left(\frac{1}{2}\right)$ -----------------→ 3 M $\theta = 50c^0$ -----------------→ 2 M
9(a)	**Find the Laplace transform of $\frac{\cos at - cosbt}{t}$** Let $f(t) = \frac{\cos at - cosb}{t}$ -----------------→ 2 M $\therefore L[f(t)] = L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = F(s)$ We know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$ -----------------→ 2 M $= \frac{1}{2}ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$ -----------------→ 3 M

9(b)	**Solve $y'' + 4y' + 3y = e^{-t}$; $y(0) = 1, y'(0) = 1$ at $t = 0$ by using Laplace transforms method.** $s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 3Y(s) = \frac{1}{s+1}$ -----------------→ 2 M $Y(s) = \frac{1}{(s+1)(s^2+4s+3)} + \frac{s+5}{(s+1)(s+3)}$ -----------------→ 1 M $Y(s) = \frac{7}{4(s+1)} + \frac{1}{2(s+1)^2} - \frac{3}{4(s+3)}$ -----------------→ 2 M Apply ILT on both sides, we get $y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$ -----------------→ 2 M
10 (a)	**Evaluate the integral $\int_0^\infty te^{-2t} sin\ 3t\ dt$ using Laplace transforms.** Let $f(t) = \sin 3t$, $L[f(t)] = \frac{3}{s^2+9} = F(s)$ -----------------→ 1 M WKT $L[tf(t)] = -\frac{d}{ds}[F(s)]$ -----------------→ 1 M $L[tsin3t] = \frac{6s}{(s^2+9)^2}$ -----------------→ 1 M $\int_0^\infty e^{-st} f(t)dt = L[f(t)]$ -----------------→ 1 M $\therefore \int_0^\infty e^{-2t} t \sin 3t\ dt = L[t \sin 3t]$ where $s = 2$ -----------------→ 1 M = 12/ 169 -----------------→ 2 M
10(b)	**Find $L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right]$ using convolution theorem** Let $\bar{f}(s) = \frac{1}{s^2+1}$; $\bar{g}(s) = \frac{1}{s^2+9}$ $L^{-1}\left[\frac{s}{(s^2+1)}\right] = \sin t$; $L^{-1}\left[\frac{s}{(s^2+9)}\right] = \frac{1}{3} sin\ 3t$ -----------------→ 2 M By convolution theorem $L^{-1}[\bar{f}(s)\bar{g}(s)] = \int_0^t f(u)g(t-u)du$ -----------------→ 1 M $\therefore L^{-1}\left[\frac{1}{(s^2+b^2)(s^2+a^2)}\right] = \frac{1}{3}\int_0^t \sin u\ \ sin3(t-u)du$ $= \frac{1}{2}\int_0^t [\cos(4u - 3t) - cos(3t - 2u)]du$ -----------------→ 2 M $= \frac{1}{24}[3\ sint - 3\sin 3t]$ -----------------→ 2 M

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